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TOTAL DOMINATION IN CLAW-FREE GRAPHS WITH MINIMUM DEGREE TWO

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Total domination in claw-free graphs with minimum degree two

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Abstract

A set S of vertices in a graph G is a total dominating set of G if every vertex of G is adjacent to some vertex in S (other than itself). The minimum cardinality of a total dominating set of G is the total domination number of G, denoted by $\gamma_t(G)$. A graph is claw-free if it does not contain $K_{1,3}$ as an induced subgraph. It is known (see J. Graph Theory 35 (2000), 21–45) that if G is a connected graph of order n with minimum degree at least two and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then $\gamma_t(G) \leq 4n/7$. In this paper, we show that this upper bound can be improved if G is restricted to be a claw-free graph. We show that every connected claw-free graph G of order n and minimum degree at least two satisfies $\gamma_t(G) \leq (n+2)/2$ and we characterize those graphs for which $\gamma_t(G) = \lfloor (n+2)/2 \rfloor$.

Keywords: bounds, claw-free graphs, total domination
AMS subject classification: 05C69

Résumé

Un ensemble S de sommets d'un graphe G est un dominant total de G si tout sommet de G est adjacent à un sommet de S (différent de lui-même). Le cardinal minimum d'un dominant total est noté $\gamma_t(G)$. Nous montrons que tout graphe connexe d'ordre n, de degré minimum $\delta \geq 2$ et sans $K_{1,3}$ induit vérifie $\gamma_t(G) \leq (n+2)/2$ et nous caractérisons les graphes pour lesquels $\gamma_t(G) = \lfloor (n+2)/2 \rfloor$.

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1 Introduction

Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2] and is now well studied in graph theory (see, for example, [1, 3, 6]). The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [4, 5].

A *total dominating set* of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S (other than itself). Every graph without isolated vertices has a total dominating set, since $S = V(G)$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. A total dominating set of G of cardinality $\gamma_t(G)$ we call a $\gamma_t(G)$-set.

For notation and graph theory terminology we in general follow [4]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A *clique* in G is a complete subgraph in G.

A *cycle* on n vertices is denoted by C_n. The minimum degree among the vertices of G is denoted by $\delta(G)$. We shall denote the set of all vertices in G of degree 2 by $S_2(G)$, or simply by S_2 if the graph G is clear from context.

For $k \geq 1$ an integer, the *k-corona* of a graph H is the graph of order $(k+1)|V(H)|$ obtained from H by attaching a path of length k to each vertex of H so that the resulting paths are vertex disjoint.

A graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. An excellent survey of claw-free graphs has been written by Flandrin, Faudree, and Ryjáček [7].

In this paper we show that every connected claw-free graph G of order n and $\delta(G) \geq 2$ satisfies $\gamma_t(G) \leq (n+2)/2$ with equality if and only if G is a cycle of length congruent to 2 modulo 4. A characterization of the connected claw-free graphs G of order n and $\delta(G) \geq 2$ satisfying $\gamma_t(G) = (n+1)/2$ is obtained.

2 Total Domination in Graphs

The total domination number of a cycle is easy to compute.

Proposition 1 ([6]) *For* $n \geq 3$, $\gamma_t(C_n) = n/2$ *if* $n \equiv 0 \pmod 4$ *and* $\gamma_t(C_n) = \lceil (n+1)/2 \rceil$ *otherwise.*

The decision problem to determine the total domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the total domination number of a graph. Cockayne et al. [2] obtained the following upper bound on the total domination number of a connected graph in terms of the order of the graph.

Theorem 2 ([2]) *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.*

Brigham, Carrington, and Vitray [1] obtained the following characterization of connected graphs of order at least 3 with total domination number exactly two-thirds their order.

Theorem 3 ([1]) *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if G is C_3, C_6 or the 2-corona of some connected graph.*

If we restrict the minimum degree to be at least two, then the upper bound in Theorem 2 can be improved.

Theorem 4 ([6]) *If G is a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then $\gamma_t(G) \leq 4n/7$.*

Favaron, Henning, Mynhardt, and Puech [3] showed that if G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq 7n/13$ and conjectured that this upper bound can be improved to $n/2$ and showed infinite families of connected cubic graphs with total domination number half their order. This conjecture was recently proven by Lam and Wei [8] who defined an **M-graph** to be a graph G with $\delta(G) \geq 2$ satisfying the condition that if $S_2 \neq \emptyset$, then the length of a longest path in $G[S_2]$ is at most one. The following beautiful result is proven in [8].

Theorem 5 ([8]) *If G is an M-graph, then $\gamma_t(G) \leq n/2$.*

Since any graph with minimum degree at least three is an M-graph, Theorem 5 immediately implies the conjecture due to Favaron et al. [3] that every graph with minimum degree at least three has total domination number at most half its order.

3 The Family $\mathcal{G}^*$

In this section, we construct an infinite family $\mathcal{G}^*$ of connected, claw-free graphs G of order n satisfying $\gamma_t(G) = (n+1)/2$.

Let $G_1, G_2, \ldots, G_7$ be the six graphs shown in Figure 1, and let $\mathcal{G} = \{G_1, G_2, \ldots, G_7\}$.

The following result is straightforward to verify.

Observation 6 *Let $G \in \mathcal{G}$ have order n. Then G is a connected claw-free graph with $\delta(G) = 2$ satisfying $\gamma_t(G) = (n+1)/2$. Furthermore, every vertex v of G, except for a neighbor of the vertex of degree 4 in G_5 and a neighbor of one of the two vertices of degree 3 in G_6 or G_7 that are incident with a bridge, belongs to a dominating set D of G such that $|D| = (n-1)/2$ and v is the only isolated vertex in $G[D]$.*

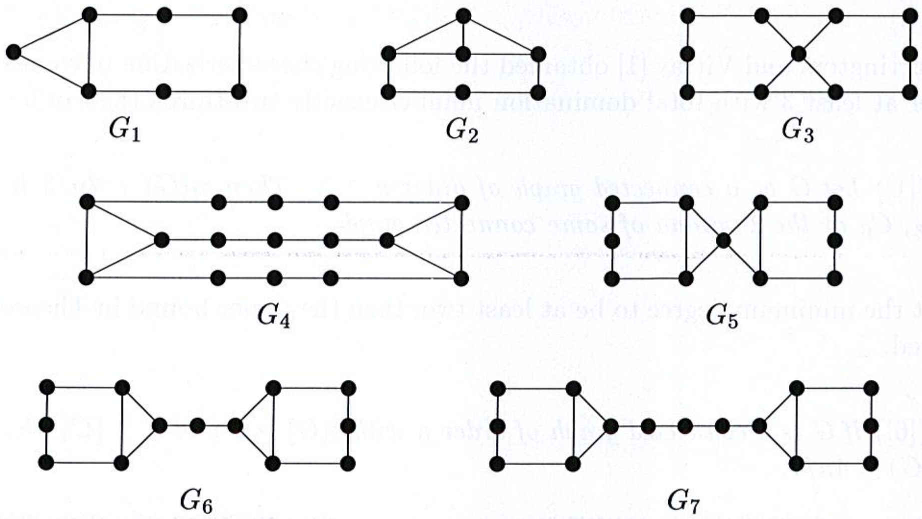

Figure 1: The family $\mathcal{G} = \{G_1, G_2, \ldots, G_7\}$.

We define an *elementary* 4-*subdivision* of a nonempty graph G as a graph obtained from G by subdividing some edge four times. A 4-*subdivision* of G is a graph obtained from G by a succession of elementary 4-subdivisions (including the possibility of none).

We shall need the following lemma from [6].

Lemma 7 ([6]) *Let G be a connected nontrivial graph and let G' be obtained from G by an elementary 4-subdivision. Then $\gamma_t(G') = \gamma_t(G) + 2$.*

We define a *good edge* of a graph G to be an edge uv in G such that both $N[u]$ and $N[v]$ induce a clique in $G - uv$. Further, we define a *good* 4-*subdivision* of G to be a 4-subdivision of G obtained by a sequence of elementary 4-subdivisions of good edges (at each stage in the resulting graph). The following observation is immediate.

Observation 8 *Let G be a claw-free graph and let G' be obtained from G by an elementary 4-subdivision of an edge e of G. Then G' is claw-free if and only if e is a good edge of G.*

For $i = 1, 2, \ldots, 7$, let $\mathcal{G}_i^* = \{G \mid G \text{ is a good 4-subdivision of } G_i\}$. We now define our family $\mathcal{G}^*$ by

$$\mathcal{G}^* = \bigcup_{i=1}^{7} \mathcal{G}_i^*.$$

It follows from Observation 8 and by the way in which the family $\mathcal{G}^*$ is constructed, that each graph $G \in \mathcal{G}^*$ is claw-free. The following result now follows readily from Observation 6 and the proof of Lemma 7 presented in [6].

Observation 9 *Let $G \in \mathcal{G}^*$ have order n. Then G is a connected claw-free graph with $\delta(G) = 2$ satisfying $\gamma_t(G) = (n+1)/2$. Furthermore, every vertex v of G, except for a neighbor of the vertex of degree 4 in G_5^* and a neighbor of one of the two vertices of degree 3 in G_6^* or G_7^* that are incident with a bridge, belongs to a dominating set D of G such that $|D| = (n-1)/2$ and v is the only isolated vertex in $G[D]$.*

4 Main Result

If we restrict G to be a connected claw-free graph, then the upper bound of Theorem 2 cannot be improved since the 2-corona of a complete graph is claw-free and has total domination number two-thirds its order. Furthermore, with this restriction on G, the upper bound of Theorem 5 cannot be improved since the graph obtained from $m \geq 2$ disjoint copies of $K_4 - e$ by selecting one vertex of degree 2 in each copy and forming a clique on the resulting set of m selected vertices is a connected, claw-free M-graph with total domination number one-half its order.

Our aim in this paper is twofold: First to show that the upper bound of Theorem 4 can be improved if we restrict G to be a claw-free graph, and, secondly, to characterize the extremal graphs achieving the new upper bound.

We will refer to a graph G as a **reduced graph** if G has no induced path on six vertices, the internal vertices of which have degree 2 in G.

We shall prove:

Theorem 10 *If G is a connected reduced claw-free graph of order n with $\delta(G) \geq 2$, then $\gamma_t(G) \leq n/2$ unless $G \in \{C_3, C_5, C_6\} \cup \mathcal{G}$.*

As an immediate consequence of Lemma 7, Observation 8 and Theorem 10 we have the following result.

Corollary 11 *If G is a connected claw-free graph of order n with $\delta(G) \geq 2$, then either*
(i) *$\gamma_t(G) \leq n/2$, or*
(ii) *G is an odd cycle or $G \in \mathcal{G}^*$, in which case $\gamma_t(G) = (n+1)/2$, or*
(iii) *$G = C_n$ where $n \equiv 2 \pmod 4$, in which case $\gamma_t(G) = (n+2)/2$.*

5 Proof of Theorem 10

We proceed by induction on the order $n \geq 3$ of a connected reduced claw-free graph G with $\delta(G) \geq 2$. If $n = 3$, then $G = C_3$ and $\gamma_t(G) = 2 = (n+1)/2$. If $n = 4$, then C_4 is a subgraph of G, and so $\gamma_t(G) = 2 = n/2$. If $n = 5$, then, by Theorem 4, either $G = C_5$, in which case $\gamma_t(G) = 3 = (n+1)/2$, or $\gamma_t(G) = 2 = (n-1)/2$. If $n = 6$, then, by Theorem 4, either $G = C_6$, in which case $\gamma_t(G) = 4 = (n+2)/2$, or $\gamma_t(G) \le 3 = n/2$. This establishes the base cases.

Suppose then that the result is true for every connected reduced claw-free graph of order less than n, where $n \ge 7$. Let G be a connected reduced claw-free graph of order n with $\delta(G) \ge 2$. If G is a cycle, then the desired result follows from Proposition 1. Hence we may assume that $G[S_2]$ is a disjoint union of paths. If G is an M-graph, then $\gamma_t(G) \le n/2$ by Theorem 5. Hence we may assume that $G[S_2]$ contains a path of length at least two. Among all paths in G, every internal vertex of which belongs to S_2, let $x_0, x_1, \dots, x_k$ be chosen so that

(i) k is as large as possible, and subject to (i),
(ii) $x_0x_k \notin E(G)$ if possible.

Hence, $\deg_G x_0 \ge 3$ and $\deg_G x_k \ge 3$ while $\deg_G x_i = 2$ for $i \in \{1, \dots, k-1\}$. Since $G[S_2]$ contains a path of length at least two, $k \ge 4$. On the other hand, since G is a reduced graph, $k \le 5$.

Let $R = N(x_0) - \{x_1\}$ and let $T = N(x_k) - \{x_{k-1}\}$. Since G is claw-free, $x_0 \ne x_k$ and each of R and T induces a clique.

Claim 1 *If $k = 5$, then $\gamma_t(G) \le n/2$ or $G = G_1$.*

Proof. Since G is a reduced graph, $x_0x_5 \in E(G)$. Since G is claw-free, the cliques $G[R]$ and $G[T]$ are the same, i.e., $R = T$. Let $G' = G - \{x_1, x_2, x_3, x_4\}$. Then G' is a connected claw-free graph of order $n' = n - 4$ with $\delta(G') \ge 2$. Since each of x_0 and x_5 lies in a triangle in G', G' is not a cycle unless $G' = K_3$ in which case $G = G_1$. Hence we may assume that G is not a cycle. Further, since G' has at least two vertices, namely x_0 and x_5, whose closed neighborhoods induce a clique, $G \notin \mathcal{G}^*$. Let D' be a $\gamma_t(G')$-set. By the inductive hypothesis, $|D'| \le n'/2 = (n-4)/2$. The set $D' \cup \{x_2, x_3\}$ is a total dominating set of G, and so $\gamma_t(G) \le |D'| + 2 \le n/2$. □

Claim 2 *If $k = 4$ and $x_0x_4 \notin E(G)$, then $\gamma_t(G) \le n/2$ or $G = \{G_2, G_3, G_4, G_7\}$.*

Proof. Let $G' = G - \{x_1, x_2, x_3, x_4\}$. Then G' is a claw-free graph of order $n' = n - 4$. Since x_0 lies in a triangle in G', G' is not a cycle unless $G' = K_3$ in which case $G = G_2$. Hence we may assume that G' is not a cycle.

Suppose $G' \in \mathcal{G}^*$. Since $N[x_0]$ induces a clique in G', it follows that $G' = G_1^*$ and that x_0 is the vertex of degree 2 in the triangle in G'. Since $G'[T]$ a clique and $x_0x_4 \notin E(G)$, $|T| = 2$ and the two vertices of T are adjacent. Hence, since G is a reduced claw-free graph, it follows that either $G = G_3$ (if $G' = G_1$ and T consists of a neighbor of x_0 in G' and a vertex at distance 2 from x_0 in G') or $G = G_4$ (if G' is an elementary 4-subdivision of one good edge of G_1 and T consists of the two vertices at distance 5 from x_0 in G') or $\gamma_t(G) \le (n-1)/2$. Hence we may assume that $G' \notin \mathcal{G}^*$.

Suppose G' is connected and $\delta(G') \geq 2$. Let D' be a $\gamma_t(G')$-set. By the inductive hypothesis, $|D'| \leq n'/2 = (n-4)/2$. The set $D' \cup \{x_2, x_3\}$ is a total dominating set of G, and so $\gamma_t(G) \leq |D'| + 2 \leq n/2$. Hence we may assume that G' is disconnected or $\delta(G') = 1$. Note that if $\delta(G') = 1$, then $|T| = 2$ and the two vertices of T are the only possible vertices of degree 1 in G', while if G' is disconnected, then since each of R and T induces a clique, $R \cap T = \emptyset$.

Let F be obtained from G' by adding all edges between x_0 and vertices in T that are not adjacent to x_0. Then F is a connected claw-free graph of order $n' = n - 4$ with $\delta(F) \geq 2$. Since $\deg_F x_0 \geq 4$, F is not a cycle.

Suppose $F \in \mathcal{G}^*$. Since the subgraph induced by $N(x_0)$ in F consists of two (disjoint) cliques each of order at least 2, it follows that $F \in \{G_3, G_5\}$ and that x_0 is the vertex of maximum degree 4 in F. If $F = G_3$, then G' is a connected graph with $\delta(G') = 2$, a contradiction. Hence $F = G_5$, and therefore $G = G_7$. Thus we may assume that $F \notin \mathcal{G}^*$.

Let S be a $\gamma_t(F)$-set. By the inductive hypothesis, $|S| \leq n'/2 = (n-4)/2$. If $S \cap (T \cup \{x_0\}) = \emptyset$, then let $D = S \cup \{x_2, x_3\}$. If $x_0 \in S$ and $S \cap T \neq \emptyset$, then let $D = S \cup \{x_1, x_4\}$. If $x_0 \in S$ and $S \cap T = \emptyset$, then let $D = S \cup \{x_3, x_4\}$. If $x_0 \notin S$ and $S \cap T \neq \emptyset$, then let $D = S \cup \{x_1, x_2\}$. In all cases, D is a total dominating set of G, and so $\gamma_t(G) \leq |D| = |S| + 2 \leq n/2$. □

By Claims 1 and 2, we may assume that $k = 4$ and $x_0x_4 \in E(G)$. Since G is claw-free, the cliques $G[R]$ and $G[T]$ are the same (and $\deg_G x_0 = \deg_G x_4$).

Claim 3 *If* $\deg_G x_0 \geq 4$, *then* $\gamma_t(G) \leq n/2$.

Proof. Let $G' = G - \{x_1, x_2, x_3, x_4\}$. Then, G' is a connected, claw-free graph of order $n' = n - 4$ with $\delta(G') \geq 2$. If G' is a cycle, then since x_0 lies in a triangle in G', $G' = K_3$ and so $\gamma_t(G) = 3 = (n-1)/2$. Hence we may assume that G' is not a cycle. Suppose $G' \in \mathcal{G}^*$. Since $N[x_0]$ induces a clique in G', and since $R = T$, it follows that $G' = G_1$ and that x_0 is the vertex of degree 2 in the triangle in G'. But then $\gamma_t(G) = 5 = (n-1)/2$. Hence we may assume that $G' \notin \mathcal{G}^*$. Let D' be a $\gamma_t(G')$-set. By the inductive hypothesis, $|D'| \leq n'/2 = (n-4)/2$. Then, $D' \cup \{x_2, x_3\}$ is a total dominating set of G, and so $\gamma_t(G) \leq |D'| + 2 \leq n/2$. □

By Claim 3, we may assume that $R = T = \{y\}$. If $\deg_G y = 2$, then $n = 6$ and $\gamma_t(G) = 3 = n/2$. Hence we may assume $\deg_G y \geq 3$. Let $Y = N(y) - \{x_0, x_4\}$. Since G is claw-free, Y induces a clique. Let $X = \{x_0, x_1, x_2, x_3, x_4\}$.

Claim 4 *If every vertex of* Y *has degree at least* 3 *in* G *(in particular, if* $\deg_G y \geq 5$*), then* $\gamma_t(G) \leq n/2$ *or* $G = G_5$.

Proof. Let $G' = G - (X \cup \{y\})$. Then G' is a connected, claw-free graph of order $n' = n - 6$ with $\delta(G') \geq 2$.

Suppose G' is a cycle. Then, since $G[Y]$ is a clique, $|Y| \in \{1,2,3\}$. By our choice of k, G' has length at most 5. If $G' = C_3$, then $\gamma_t(G) = 4 = (n-1)/2$ (irrespective of whether $|Y| = 1$ or $|Y| = 2$ or $|Y| = 3$). If $G' = C_4$, then since G is claw-free, $|Y| = 2$ and $\gamma_t(G) = 5 = n/2$. If $G' = C_5$, then $G \in G_5$. Hence we may assume that G' is not a cycle.

Suppose $G' \in \mathcal{G}^*$. Let $v \in Y$. Since $G'[Y]$ is a clique, and since G is claw-free, it follows that we can choose v so that it is neither a neighbor of the vertex of degree 4 in G_5^* nor a neighbor of a vertex of degree 3 in G_6^* or G_7^* that is incident to a bridge. Hence by Observation 9, there exists a dominating set D of G' such that $v \in D$, $|D| = (n'-1)/2 = (n-7)/2$ and v is the only isolated vertex in $G'[D]$. Thus, $D \cup \{x_2, x_3, y\}$ is a total dominating set of G, and so $\gamma_t(G) \leq |D| + 3 = (n-1)/2$. Hence we may assume that $G' \notin \mathcal{G}^*$.

Let D' be a $\gamma_t(G')$-set. By the inductive hypothesis, $|D'| \leq n'/2 = (n-6)/2$. The set $D' \cup \{x_0, x_1, x_4\}$ is a total dominating set of G, and so $\gamma_t(G) \leq |D'| + 3 \leq n/2$. □

By Claim 4, we may assume that $\deg_G y = 3$ or 4 and at least one neighbor z of y has degree 2. Let $N(z) - \{y\} = \{t\}$ (the edge ty may or may not exist). If $n = 8$, then $ty \in E(G)$ and $\{x_0, x_1, x_4, y\}$ is a total dominating set of G, and so $\gamma_t(G) \leq 4 = n/2$. So we may suppose $n \geq 9$.

Claim 5 *If all the vertices of $N(t) - \{y, z\}$ have degree at least 3 in G, then $\gamma_t(G) \leq n/2$ or $G = G_6$.*

Proof. Let $G' = G - (X \cup \{t, y, z\})$. Then G' is a connected, claw-free graph of order $n' = n - 8$ with $\delta(G') \geq 2$.

Suppose G' is a cycle. Since G is claw-free, no neighbor of y belongs to G'. Further, t has exactly two neighbors on the cycle and these two neighbors are adjacent. Hence, by our choice of k, G' has length at most 5. If $G' = C_3$, then $\gamma_t(G) = 5 = (n-1)/2$. If $G' = C_4$, then $\gamma_t(G) = 6 = n/2$. If $G' = C_5$ and $ty \in E(G)$, then $\gamma_t(G) = 6 = (n-1)/2$. If $G' = C_5$ and $ty \notin E(G)$, then $G = G_6$. Hence we may assume that G' is not a cycle.

Suppose $G' \in \mathcal{G}^*$. Let $v \in N(t) \cap V(G')$. Since $N(t) \cap V(G')$ is a clique, and since G is claw-free, it follows that we can choose v so that it is neither a neighbor of the vertex of degree 4 in G_5^* nor a neighbor of a vertex of degree 3 in G_6^* or G_7^* that is incident to a bridge. Hence by Observation 9, there exists a dominating set D of G' such that $v \in D$, $|D| = (n'-1)/2 = (n-9)/2$ and v is the only isolated vertex in $G'[D]$. Thus, $D \cup \{x_0, x_1, x_4, t\}$ is a total dominating set of G, and so $\gamma_t(G) \leq |D| + 4 = (n-1)/2$. Hence we may assume that $G' \notin \mathcal{G}^*$.

Let D' be a $\gamma_t(G')$-set. By the inductive hypothesis, $|D'| \leq n'/2 = (n-8)/2$. The set $D' \cup \{x_1, x_2, y, z\}$ is a total dominating set of G (irrespective of whether the edge ty is present or not). Hence, $\gamma_t(G) \leq |D'| + 4 \leq n/2$. □

By Claim 5, we may assume that $N(t) - \{y, z\}$ contains a vertex u of degree 2 in G. Let $N(u) - \{t\} = \{w\}$ (the edge tw may or may not exist). By the claw-freeness of G, the only neighbors of t are u and z and possibly y and w.

Claim 6 *If $ty \in E(G)$, then $\gamma_t(G) \le n/2$.*

Proof. Let $G' = (G - \{t, z\}) + uy$. Then G' is a connected, claw-free graph of order $n' = n-2$ and G' is not a cycle.

Suppose $G' \in \mathcal{G}^*$. Then $G' = G_6$ where u is the vertex of G' incident with two bridges (note that the case that G' is obtained from G_6 an elementary 4-subdivision of a bridge of G_6 where u is a vertex of degree 2 in G' incident with a bridge and adjacent to a vertex of degree 3 cannot occur by our choice of k). But then $\gamma_t(G) = 7 = (n-1)/2$. Hence we may assume that $G' \notin \mathcal{G}^*$.

If $\delta(G') = 1$, then $N(w) = \{u, t\}$, and so $n = 10$ and $\gamma_t(G) \le |\{x_1, x_2, t, y\}| = 4 = (n-2)/2$. Hence we may assume $\delta(G') \ge 2$. Let D' be a $\gamma_t(G')$-set. By the inductive hypothesis, $|D'| \le n'/2 = (n-2)/2$. If $\{u, y\} \cap D' \ne \emptyset$, let $D = D' \cup \{t\}$. If $\{u, y\} \cap D' = \emptyset$, let $D = D' \cup \{y\}$ (note that in order to dominate y, at least one of x_0 and x_4 is in D'). In any case, D is a total dominating set of G, and so $\gamma_t(G) \le |D| = |D'| + 1 \le n/2$. □

By Claim 6, we may assume that $ty \notin E(G)$. Therefore, since $yw \notin E(G)$, tw must be an edge of G by condition (ii) in the choice of the path $x_0, x_1, \ldots, x_k$.

If $\deg_G w = 2$, then $n = 10$ and $\gamma_t(G) \le |\{x_1, x_2, t, y, z\}| = 5 = n/2$. Hence we may assume $\deg_G w \ge 3$. Let $W = N(w) - \{u, t\}$. Since G is claw-free, $G[W]$ is a clique.

Let G' be obtained from $G - \{t, u, w, z\}$ by adding all edges between y and vertices in W. Then G' is a connected, claw-free graph of order $n' = n - 4$ with $\delta(G') \ge 2$ and G' is not a cycle.

Suppose $G' \in \mathcal{G}^*$. Then either $G = G_5$ or $G = G_6$ or $G = G_7$. If $G = G_5$ (resp., $G = G_6$ or $G = G_7$), then the set $\{t, w, x_0, x_1, x_4\}$ can be extended to a total dominating set of G by adding two (resp., three or four) additional vertices, and so $\gamma_t(G) \le (n-1)/2$. Hence we may assume that $G' \notin \mathcal{G}^*$.

Let D' be a $\gamma_t(G')$-set. By the inductive hypothesis, $|D'| \le n'/2 = (n-4)/2$. If $y \notin D'$, let $D = D' \cup \{t, z\}$. If $y \in D'$ and $D' \cap W \ne \emptyset$, let $D = D' \cup \{w, z\}$. If $y \in D'$ and $D' \cap W = \emptyset$, let $D = D' \cup \{t, w\}$. In any case, D is a total dominating set of G, and so $\gamma_t(G) \le |D| = |D'| + 2 \le n/2$, as desired. This completes the proof of the theorem. □.

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