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LES ARBRES DE CRISTAL

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Résumé : Considérons un graphe non orienté et connexe $G = (V, E)$ ayant pour l'ensemble des nœuds V et l'ensemble des arêtes E, avec $c_{uv} > 0$ étant l'énergie associée à une arête $uv \in E$. Soit T un arbre couvrant de G dont la racine est un noeud distingué $k \in V$, où chaque noeud $v \in V - \{k\}$ accumule les informations sur l'énergie des arêtes dans le chemin du noeud racine k jusqu'à v en T. Nous disons que deux noeuds u et v présentent des potentiels d'équilibre stable en T si l'arête $uv \in E$ qui n'est pas en T a plus d'énergie que toute autre arête dans le chemin direct de u à v en T. Deux noeuds u et v, avec $uv \in E$ et $uv \notin T$, ont un potentiel d'équilibre instable en T si les chemins de u et v à leur premier ancêtre commun dans leurs chemins à k contiennent exactement (chaque trajet) la même quantité d'arêtes d'énergies de valeurs plus grandes que celle de uv et toutes les autres arêtes du chemin direct de u à v en T ont une valeur d'énergie qui ne dépasse pas celle de uv. Un arbre couvrant k-enraciné de G avec des potentiels équilibrés est nommé un arbre de cristal. Dans ce travail, nous présentons quelques propriétés des arbres de cristal et discutons leurs relations / différences par rapport aux arbres couvrants d'énergie minimale et aux arbres k-enracinés de plus court chemin. Nous introduisons une nouvelle fonction de potentiel permettant de décrire l'ensemble des arbres de cristal d'un graphe G comme étant des solutions d'un système linéaire polynomial en nombre des contraintes et des variables.
Mots-clés: arbre de cristal, arbre couvrant de poids minimum, fonction de modélisation de super-ensemble.

$\mathcal{C}$rystal $\mathcal{T}$rees

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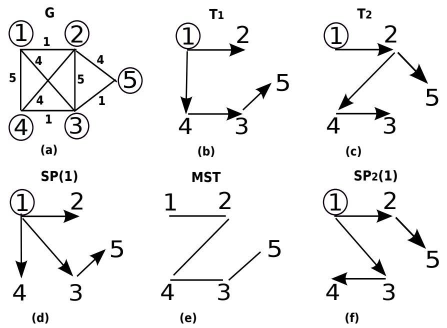
Abstract. Consider an undirected and connected graph $G = (V, E)$ of node set V and edge set E, with $c_{uv} > 0$ being the energy associated with an edge $uv \in E$. Let T be a pending spanning tree of G rooted at a distinguished node $k \in V$, where each node $v \in V - \{k\}$ accumulates the information of the energy of the edges in the path from the root node k until v in T. We say that two nodes u and v present stable potential equilibrium in T if any edge $uv \in E$ not in T have greater energy than the edge with larger energy in the direct path from u to v in T. Two nodes u and v, with $uv \in E$ and $uv \notin T$, have instable potential equilibrium in T if the paths from u and v to their first common ancestor in their paths to k contain exactly (each path) the same edge energies having larger values than the one of uv and all the remaining edges in the direct path from u to v in T have not greater energy than c_{uv}. A k-rooted pending spanning tree of G with equilibrated potentials is a crystal tree. In this work we present some properties of crystal trees and discuss their relations/differences with respect to spanning trees of minimum energy and k-rooted shortest path trees. We introduce a potential function allowing to describe the complete set of crystal trees of a graph G as solutions of a polynomially bounded system of linear inequalities.

Keywords: crystal tree, minimum spanning tree, multi-set modeling function.

1 Introduction

Let $G = (V, E)$ be an undirected and connected graph (V and E are the sets of nodes and edges, respectively) where with each edge $uv \in E$ we associate an energy $c_{uv} > 0$. Let T be a pending spanning tree of G rooted at a distinguished node $k \in V$, where each node $v \in V - \{k\}$ accumulates the information (as a multi-set of values, in a mathematical sense) of the energy of the edges in the path from the root node k until v in T. We represent the accumulated potential information of a node v in T by a multi-set Φ_v^k, with the root node having null potential (i.e. $\Phi_k^k = \emptyset$). Thus, if $uv \in E$ is a k-rooted pending spanning tree T of G and u precedes v in the path from k to v, then $\Phi_v^k = \Phi_u^k \cup \{c_{uv}\}$. We say that two nodes u and v present stable potential equilibrium in T if any edge $uv \in E$ not in T have greater energy than the edge with larger energy in the direct path from u to v in T. Two nodes u and v, with $uv \in E$ and $uv \notin T$, have instable potential equilibrium in T if the paths from u and v to their first common ancestor in their paths to k contain exactly (each path) the same edge energies having larger values than the one of uv and all the remaining edges in the direct path from u to v in T have not greater energy than c_{uv}. A k-rooted pending spanning tree of G with equilibrated (i.e. stable or instable) potentials is a crystal tree. In Figure 1 we show some examples to help understanding the concept of a crystal tree.

Fig. 1. In (a) we have a graph G. In (b) and (c) we have two 1-rooted pending spanning trees T_1 and T_2 of G. We use arcs only to structure these pending trees from the circled root nodes. T_1 is not a crystal tree. Indeed, edge $(2,4) \notin T_1$ has energy $c_{24} < c_{14}$ and there is no edge with energy equal to c_{14} in the path from 2 to 1, thus the potentials Φ_2^1 and Φ_4^1 of T_1 are not equilibrated. In the other hand, edge $(3,5) \notin T_2$ has energy $c_{35} < c_{25}$. However, the number of edges with energy equal to c_{25} are the same in the paths from 5 to 2 and from 3 to 2 (node 2 is the first common ancestor in the paths from nodes 5 and 3 to the root node 1). In this case, the potentials $\Phi_5^1 = \{1,4\}$ and $\Phi_3^1 = \{1,4,1\}$ are in instable equilibrium. As the remaining edges not in T_2 have larger energy than the ones in this tree, we say that T_2 is a crystal tree of G. In (d) and (f) we have two shortest-paths (in energy) trees $SP(1)$ and $SP_2(1)$ rooted at node 1. Note that $SP(1)$ is not a crystal tree, while $SP_2(1)$ is crystal. In (e) we have a minimum (in energy) spanning tree MST of G. For every choice of the root node for this tree, all the resulting structures are crystal trees.



In this work we present some properties of crystal trees and discuss relations and differences with respect to spanning trees of minimum energy and k-rooted shortest (in energy) path trees. We introduce a mathematical function that models the node potential multi-sets, thus allowing to describe the complete set of crystal trees of a graph G as solutions of a polynomially bounded system of linear inequalities.

The technique we develop for representing the new class of crystal trees is novel for the domain of graph theory and has important applications in the field of combinatorial optimization. Indeed, the robust tree problem (see e.g. [3]) asks for solutions presenting, besides the main purpose of this problem, a minimum spanning tree structure that is presented here as a particular case of crystal trees.

2 Algebraic multi-set modeling system

Given any spanning tree T of a connected graph $G = (V, E)$, knowing if T is a crystal tree with respect to a given root node $k \in V$ can be done in polynomial time. For such, we need just to realize some search operations in paths. If we need to exhibit one such tree, any minimum spanning tree greed algorithm [1] resolves this task for crystal trees having only node potentials in stable equilibrium. However, it seems hard to look for crystal trees presenting node potentials in instable equilibrium. We do not know if it is possible conceiving an algorithm that does this task in polynomial time, even if we use a spanning tree of minimum energy as starting point. Nevertheless, there is a mathematical way to represent the complete set of crystal trees as solutions of a system of linear inequalities.

Initially, consider a formal definition of a crystal tree. Let T be a k-rooted pending spanning tree of G having null potential (i.e. $\Phi_k^k = \emptyset$). Consider an edge (u, v) of E not in T. Let $\Phi_v^k = \{c_{kv_1}, c_{v_1v_2}, \cdots, c_{v_{j-1}v_j}, c_{v_jv_{j+1}}, \cdots, c_{v_pv}\}$ be the multi-set (potential) of v, where $k, v_1, v_2, \cdots, v_p, v$ is the sequence of nodes in the path from k to v. Let $\Phi_u^k = \{c_{ku_1}, c_{u_1u_2}, \cdots, c_{u_{j-1}u_j}, c_{u_ju_{j+1}}, \cdots, c_{u_qu}\}$ be the multi-set of u, where $k, u_1, u_2, \cdots, u_q, u$ is the sequence of nodes in the path from k to u. Assume that $v_1 = u_1,\ v_2 = u_2, \cdots, v_{j+1} = u_{j+1}$ are the nodes common to the paths from k to v and to u, with u_{j+1} being the first common ancestor in the inverse paths from v and u to k. Define, for every edge (u, v) not in T, $\hat{U} = \{(u_{j+1}, u_{j+2}), \cdots, (u_q, u)\}$ and $\hat{V} = \{(v_{j+1}, v_{j+2}), \cdots, (v_p, v)\}$ (with $\hat{U}$ and $\hat{V}$ depending on the distinct nodes u and v).

Definition 1 *T, a k-rooted pending spanning tree of $G = (V, E)$, is a crystal tree if all edges (u, v) of E not in T are such that exactly one of the two conditions is satisfied:*

1. *$c_{uv} \geq c_e$, for all $e \in \hat{U} \cup \hat{V}$.*
2. *$E_u \Delta E_v = \emptyset$, with $E_u := \{c_e \mid e \in \hat{U},\ c_{uv} < c_e\}$ and $E_v := \{c_e \mid e \in \hat{V},\ c_{uv} < c_e\}$, where E_u and E_v are multi-sets and Δ stands for the multi-set symmetric difference operation between E_u and E_v.*

Remark 1 *In the direct path between any pair of nodes u and v of a k-rooted pending spanning tree T of $G = (V, E)$, there exist at most $\min\{|V| - 1, M\}$ edges of same energy, where M is the number of occurrences of the energy value with largest occurrence among all the c_e values, for all $e \in E$.*

Definition 2 *Let $\mathcal{O} = \{o_1, o_2, \cdots, o_s\}$, with $s \leq |E|$, be the set of s distinct energy values of the edges in E. Define the expanded energy of an element i in $\mathcal{O}$ as*

$$\phi(i) = b^{|\{o \in \mathcal{O} \mid o < i\}|} \tag{1}$$

where $b = 1 + \min\{|V| - 1, M\}$ denotes the factor (base) of expansion of the energies in $\mathcal{O}$.

Definition 3 *Let T be a k-rooted pending spanning tree of $G = (V, E)$. Let the sequence of nodes in the path from k to v in T be $v_0, v_1, v_2, \cdots, v_j$, with $v_0 = k$ and $v_j = v$ for some j. Define the expanded potential of a node v in T as*

$$\Phi_v^k = \sum_{i=1}^{j} \phi(c_{i-1,i}) \tag{2}$$

with $\Phi_k^k = 0$. Definitions 2 and 3 can be used to characterize alternatively a crystal tree.

Theorem 1 *Let T be a k-rooted pending tree of $G = (V, E)$ and Φ^k, with $\Phi_k^k = 0$, be the expanded node potentials of T as in the Definition 3. T is crystal if and only if*

$$|\Phi_u^k - \Phi_v^k| \leq (b-1)\phi(c_{uv}), \quad \forall\, uv \ \in \ E \tag{3}$$

Proof. It follows from the Definition 1 of a crystal tree. Note that if the condition (2) of the Definition 1 were not satisfied (i.e. $E_u \Delta E_v \neq \emptyset$ for some pair of nodes u and v in T), there should exist at least one edge with (unbalanced) expanded energy strictly larger than the one of c_{uv}. As the number of edges with smaller expanded energy than $\phi(c_{uv})$ is at most $\min\{|V|-1, M\}$, and the base b is one unit larger than this value with M as defined above, then (3) would be violated because edges with smaller expanded energy than $\phi(c_{uv})$ cannot be used to balance the expanded potential of these nodes. □

We present now a system of linear inequalities describing the complete set of crystal trees of a graph $G = (V, E)$. For this, we need to represent any k-rooted pending spanning tree of G by an arborescence. This can be done by using one of the directed tree models in [2] exploring the idea of sending one unit of flow from the root node k to every node in $V - \{k\}$. In this case, the resulting aggregated flow leaving the root node k must be equal to $|V| - 1$ units and the resulting flow entering the remaining nodes must be equal to 1. Consider $\hat{G} = (V, A)$ the directed graph obtained from G, where $\hat{G}$ has set of nodes V and set of arcs $A := E \cup \{vu \mid uv \ \in \ E\}$, with arcs uv and vu having the same energy as the corresponding edge in E. Represent a k-rooted arborescence T of $\hat{G}$ by a node-arc incidence vector $x \in \{0,1\}^{|A|}$, where $x_{uv} = 1$ if arc uv belongs to T, and $x_{uv} = 0$, otherwise. Note that T must have exactly $|V| - 1$ arcs. Consider $f_{uv} \geq 0$ the amount of flow in arc uv and assume that if an arc is not in T, then no flow traverses this arc. Thus, the set $\mathcal{T}_k$ of k-rooted arborescences of $\hat{G}$ can be given by

$$\mathcal{T}_k = \left\{ \begin{array}{rcl} \displaystyle\sum_{uv \in A} x_{uv} & = & |V| - 1, \\ \displaystyle\sum_{j \mid kj \in A} f_{kj} & = & |V| - 1, \\ \displaystyle\sum_{i \mid iv \in A} f_{iv} - \sum_{j \mid vj \in A} f_{vj} & = & 1, \quad \forall\, v \in V - \{k\} \\ f_{uv} & \leq & (|V| - 1)x_{uv}, \quad \forall\, uv \in A \\ x \in \{0,1\}^{|A|}, \quad f \geq \mathbf{0} & & \end{array} \right\}$$

and using non-negative variables $\Phi^k \in \mathbb{R}_+^{|V|}$ to represent the expanded node potentials in T, with $\Phi_k^k = 0$, we can describe the set $\mathcal{C}_k$ of k-rooted pending crystal trees of G as

$$\Phi_v^k - \Phi_u^k \leq \phi(c_{uv}) + \mathcal{M}(1 - x_{uv}), \ \forall\, uv \in A \tag{4}$$

$$\Phi_v^k - \Phi_u^k \geq \phi(c_{uv}) - \mathcal{M}(1 - x_{uv}), \ \forall\, uv \in A \tag{5}$$

$$\Phi_v^k - \Phi_u^k \leq (b - 1)\phi(c_{uv}), \ \forall\, uv \in A \tag{6}$$

$$\Phi_k^k = 0, \quad \Phi^k \geq \mathbf{0} \tag{7}$$

$$x \in \mathcal{T}_k \tag{8}$$

where $\mathcal{M}$ is a very large (infinite) positive constant and ϕ is the expanded edge energy function as defined above. Constraints (4) and (5) impose that if an arc (u, v) is in T, then we must have $\Phi_v^k - \Phi_u^k = \phi(c_{uv})$; otherwise, both constraints are satisfied by all solutions of this system (i.e. they become $|\Phi_v^k - \Phi_u^k| \leq \infty$). Constraints (6) impose that inequality (3) of the Theorem 1 must be satisfied. To see this, just consider the two corresponding constraints for each arc uv and vu of A. Constraints (7) impose the domain of the Φ^k variables, with the expanded potential of the root node being null. Constraints (8) state that the vector of binary variables x describes a general k-rooted arborescence T of $\hat{G}$, induced by the non null entries of this vector.

Proposition 1 *The set of feasible solutions of the model* (4)-(8) *corresponds to the complete set of k-rooted crystal trees.*

Proof. Let $\bar{\Phi}^k$ and $\bar{x}$ be a feasible solution for (4)-(8). By (8), $\bar{x}$ is a k-pending spanning tree rooted at k. By (7), $\bar{\Phi}_k^k = 0$. Note that if an arc uv is in the solution, then by (4) and (5) we have $\bar{\Phi}_v^k = \bar{\Phi}_u^k + \phi(c_{uv})$. Thus, the $\bar{\Phi}^k$ variables accumulate correctly the expanded potential of any node. Constraints (6) impose that condition (3) of the Theorem 1 is satisfied, thus the resulting tree is crystal. In the other hand, every k-rooted crystal tree corresponds to a feasible solution of the above system. To see this, consider the k-rooted pending tree structure and orient its edges following the path from k to every node in $V - \{k\}$ and set the corresponding arc variables equal to one for all arcs appearing in this tree and set at zero all the remaining arc variables. Finally, set the node potential variables (by using the expanded energy function ϕ) according to the crystal tree node potentials. Clearly, this is a feasible solution for (4)-(8). □

3 Properties of crystal trees

We present below some properties of crystal trees that allow distinguishing them from k-rooted shortest paths trees and from minimum spanning trees.

Proposition 2 *Let T be a shortest path tree of $G = (V, E)$ with root node k. In this case:*

1. *T is not necessarily a crystal tree of G.*
2. *if T is a crystal tree of G, then T is not necessarily a minimum spanning tree of G.*

It is the case of the 1-rooted shortest-path (in energy) tree $SP(1)$ in the Figure 1 (d), where the nodes 1 and 4 are clearly not equilibrated. However, $SP_2(1)$ in the Figure 1 (f) is a 1-rooted shortest-path tree that is also a crystal tree: the edge $(2, 4)$ does not belong to this tree, but its energy is larger than all these ones in the paths from 2 and 4 to the root node 1; and the edge $(3, 5) \notin SP_2(1)$ has energy smaller than the ones of the edges $(1, 3)$ and $(2, 5)$ that are of same value and appear in same number (i.e. they are balanced) in the paths from 3 and 5 to the root node 1.

Proposition 3 *Let T be a k-rooted crystal tree of $G = (V, E)$ and a_{uv} denote the first common ancestor in the paths from u and v to k. If*

$$|\Phi(u) - \Phi(a_{uv})| \leq \phi(c_{uv}) \quad and \quad |\Phi(v) - \Phi(a_{uv})| \leq \phi(c_{uv}),\ \forall\, uv \in E,\ uv \notin T$$

then the crystal tree T is in stable equilibrium; otherwise, T is in instable equilibrium.

Proposition 3 is an alternative way of characterizing the two possible states of equilibrium of a crystal tree.

Corollary 1 *T is a minimum (in energy) spanning tree of $G = (V, E)$ if and only if for all $k \in V$, T is a k-rooted crystal tree.*

Proof. If T is a minimum spanning tree of $G = (V, E)$ then it follows directly from the definition of a minimum spanning tree (see e.g. [1]) that for all $k \in V$, T is a k-rooted crystal tree where all nodes are in stable equilibrium. Suppose now that for all $k \in V$, T is a k-rooted crystal tree and that T is not a spanning tree of minimum energy of G. Thus, there is a k-rooted crystal tree T of G, for some k, whose some nodes (say u and v) are in instable equilibrium. But, in this case, when T is rooted at u or at v, these two nodes are not in equilibrium with respect to T, thus contradicting the fact that T is a k-rooted crystal tree for all $k \in V$. Therefore, T must be a minimum (in energy) spanning tree of G. $\square$

Corollary 1 establishes implicitly that we need to know only the 'order' each value has in an increasing ordered list of the distinct edge energy values to determine which edges (not forming a cycle) can be present in any minimum spanning tree. This is the idea behind a greedy algorithm [1] for the problem of determining an independent set of minimum weight in the corresponding matroid structure.

Corollary 2 *If all the edges of $G = (V, E)$ have distinct energy values, then G has a unique crystal tree T that is also its minimum (in energy) spanning tree.*

Proof. It is not difficult to see that T is composed only of stable potentials, thus it is a minimum spanning tree of G. The proof of its uniqueness follows directly from the one of the minimum spanning tree of G under such assumption (see e.g. [1] for further details). $\square$

Corollary 3 *There exists a system of linear inequalities allowing to characterize all minimum spanning trees of a graph $G = (V, E)$.*

The idea is, for each $k \in V$, to use $x^k \in \{0,1\}^{|A|}$ decision variables, flow variable vectors $f^k \in \mathbb{R}_+^{|A|}$ and potential $\Phi^k \in \mathbb{R}_+^{|V|}$ variable vectors with their corresponding constraints (4)-(8), and to employ additional constraints for the $x^k \in \mathcal{C}_k$ variables in order to establish that the arcs associated with all the x^k decision variables induce the same edges appearing in any feasible solution for the resulting system. Thus, we propose the following model of linear inequalities for obtaining minimum spanning trees

$$
\begin{aligned}
\Phi_v^k - \Phi_u^k \le \phi(c_{uv}) + \mathcal{M}(1 - x_{uv}^k), \quad \forall\, uv \in A, \forall\, k \in V \qquad (9)\\
\Phi_v^k - \Phi_u^k \ge \phi(c_{uv}) - \mathcal{M}(1 - x_{uv}^k), \quad \forall\, uv \in A, \forall\, k \in V \qquad (10)\\
\Phi_v^k - \Phi_u^k \le (b-1)\phi(c_{uv}), \quad \forall\, uv \in A, \forall\, k \in V \qquad (11)\\
x_{uv}^1 + x_{vu}^1 = x_{uv}^k + x_{vu}^k, \quad \forall\, uv \in E, \forall\, k \in V - \{1\} \qquad (12)\\
\Phi_k^k = 0, \quad \Phi^k \ge \mathbf{0}, \quad \forall\, k \in V \qquad (13)\\
x^k \in \mathcal{T}_k, \quad \forall\, k \in V \qquad (14)
\end{aligned}
$$

where constraints (12) impose that we must have the same edges induced by the x^k variables to be considered with the expanded node potentials Φ^k, and the remaining constraints impose that we must have a k-rooted crystal tree for all $k \in V$. To the best of our knowledge, this is the first work modeling minimum spanning trees only by means of a system of linear inequalities.

An interesting question is knowing if we really need to use all blocks of constraints defining k-rooted crystal trees for every value of k (or only a part of them) in order to obtain a minimum spanning tree model with smaller dimensions. Moreover, as the theory we develop here requires working with very large numbers, how to overcome the limited machine technology in order to handle the expanded node potentials constraints implicitly, for instance, in a branch-and-bound algorithm. All these technical details, as well as practical experiments to show the viability of our models, constitute research works in progress.

4 Final remarks

This work introduces and characterizes the novel set of k-rooted crystal trees. Based on the original idea of the expanded potential function to model multi-sets, we present some properties of these structures and propose a linear system, whose feasible solutions are k-rooted crystal trees. This system is important because it can be used to represent spanning trees of minimum weight in undirected graphs. Thus, opening many possibilities (as future research) for solving more complex optimization problems constrained to have such a structure.

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