The Isabelle/Pure framework

Pure syntax and primitive rules

$\Rightarrow$	$::$	$(type,\ type)\ type$	function type constructor
$\bigwedge$	$::$	$(\alpha \Rightarrow prop) \Rightarrow prop$	universal quantifier
$\Longrightarrow$	$::$	$prop \Rightarrow prop \Rightarrow prop$	implication

$$\frac{\begin{array}{c}[x :: \alpha]\\ \vdots \\ b(x) :: \beta\end{array}}{\lambda x.\ b(x) :: \alpha \Rightarrow \beta}\,(\Rightarrow I) \qquad \frac{b :: \alpha \Rightarrow \beta \quad a :: \alpha}{b\ a :: \beta}\,(\Rightarrow E)$$

$$\frac{\begin{array}{c}[x]\\ \vdots \\ B(x)\end{array}}{\bigwedge x.\ B(x)}\,(\bigwedge I) \qquad \frac{\bigwedge x.\ B\ x}{B\ a}\,(\bigwedge E)$$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \Longrightarrow B}\,(\Longrightarrow I) \qquad \frac{A \Longrightarrow B \quad A}{B}\,(\Longrightarrow E)$$

Pure equality

$\equiv :: \alpha \Rightarrow \alpha \Rightarrow prop$

Axioms for $t \equiv u$: α, β, η, *refl*, *subst*, *ext*, *iff*

Unification: solving equations modulo $\alpha\beta\eta$

- Huet: full higher-order unification (infinitary enumeration!)
- Miller: higher-order patterns (unique result)

Hereditary Harrop Formulas

Define the following sets:

$\boldsymbol{x}$	variables
$\boldsymbol{A}$	atomic formulae (without $\bigwedge/\Longrightarrow$)
$\bigwedge \boldsymbol{x}^*.\ \boldsymbol{A}^* \Longrightarrow \boldsymbol{A}$	Horn Clauses
$\boldsymbol{H} \stackrel{\text{def}}{=} \bigwedge \boldsymbol{x}^*.\ \boldsymbol{H}^* \Longrightarrow \boldsymbol{A}$	Hereditary Harrop Formulas (HHF)

Conventions for results:

- outermost quantification $\bigwedge x.\ B\ x$ is rephrased via schematic variables $B\ ?x$
- equivalence $(A \Longrightarrow (\bigwedge x.\ B\ x)) \equiv (\bigwedge x.\ A \Longrightarrow B\ x)$ produces canonical HHF

Representing Natural Deduction rules

Examples:

$$\frac{P \quad Q}{P \wedge Q} \qquad \bigwedge P\ Q.\ P \Longrightarrow Q \Longrightarrow P \wedge Q$$

$$\frac{\begin{array}{c}[P] \\ \vdots \\ Q\end{array}}{P \rightarrow Q} \qquad \bigwedge P\ Q.\ (P \Longrightarrow Q) \Longrightarrow P \rightarrow Q$$

$$\frac{P\ 0 \quad \begin{array}{c}[n][P\ n] \\ \vdots \\ P\ (Suc\ n)\end{array}}{P\ n} \qquad \bigwedge P\ n.\ P\ 0 \Longrightarrow (\bigwedge n.\ P\ n \Longrightarrow P\ (Suc\ n)) \Longrightarrow P\ n$$

Rule composition (back-chaining)

$$\frac{\overline{A} \Longrightarrow B \quad B' \Longrightarrow C \quad B\,\theta = B'\theta}{\overline{A}\,\theta \Longrightarrow C\,\theta}\,(compose)$$

$$\frac{\overline{A} \Longrightarrow B}{(\overline{H} \Longrightarrow \overline{A}) \Longrightarrow (\overline{H} \Longrightarrow B)}\,(\Longrightarrow\text{-}lift)$$

$$\frac{\overline{A}\ \overline{a} \Longrightarrow B\ \overline{a}}{(\bigwedge \overline{x}.\ \overline{A}\ (\overline{a}\ \overline{x})) \Longrightarrow (\bigwedge \overline{x}.\ B\ (\overline{a}\ \overline{x}))}\,(\bigwedge\text{-}lift)$$

General higher-order resolution

$$\frac{\begin{array}{rl} \textit{rule}\colon & \overline{A}\ \overline{a} \Longrightarrow B\ \overline{a} \\ \textit{goal}\colon & (\bigwedge \overline{x}.\ \overline{H}\ \overline{x} \Longrightarrow B'\ \overline{x}) \Longrightarrow C \\ \textit{goal unifier}\colon & (\lambda \overline{x}.\ B\ (\overline{a}\ \overline{x}))\ \theta = B'\theta \end{array}}{(\bigwedge \overline{x}.\ \overline{H}\ \overline{x} \Longrightarrow \overline{A}\ (\overline{a}\ \overline{x}))\ \theta \Longrightarrow C\ \theta}\ (\textit{resolution})$$

$$\frac{\begin{array}{rl} \textit{goal}\colon & (\bigwedge \overline{x}.\ \overline{H}\ \overline{x} \Longrightarrow A\ \overline{x}) \Longrightarrow C \\ \textit{assm unifier}\colon & A\ \theta = H_i\ \theta\ \ \text{(for some } H_i\text{)} \end{array}}{C\ \theta}\ (\textit{assumption})$$

Both inferences are omnipresent in Isabelle/Isar:

- $\textit{resolution}$: e.g. $\textit{OF}$ attribute, $\textit{rule}$ method, **also** command
- $\textit{assumption}$: e.g. $\textit{assumption}$ method, implicit proof ending